Math 255A Lecture 2 Notes

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1 Dual Spaces and the Geometric Hahn-Banach Theorem

1.1 The dual space

Last time, we established the analytic version of the Hahn-Banach theorem. Given Banach spaces B_1, B_2, let $\mathcal{L}(B_1, B_2)$ be the space of continuous linear maps $T : B_1 \to B_2$. Then $\mathcal{L}(B_1, B_2)$ is a Banach space when equipped with the norm

$$\|T\| = \sup_{0 \neq x \in B_1} \frac{\|Tx\|_{B_2}}{\|x\|_{B_1}}.$$

Remark 1.1. To get that $\mathcal{L}(B_1, B_2)$ is complete, we only need that B_2 is complete.

Here is a special case of this construction.

Definition 1.1. Let B be a complex Banach space. The **dual space** $B^* = \mathcal{L}(B, \mathbb{C})$ is the space of linear continuous forms on B.

When $x \in B$ and $\xi \in B^*$, write $\langle x, \xi \rangle := \xi(x)$ so that the form $(x, \xi) \mapsto \langle x, \xi \rangle$ on $B \times B^*$ is bilinear.

Example 1.1. Let $B = L^1(\mathbb{R})$. Then $B^* = L^\infty(\mathbb{R})$. We claim that there exists a continuous linear form on $L^\infty(\mathbb{R})$ which is not of the form $u \mapsto \langle f, u \rangle = \int f u \, dx$. Indeed, by the Hahn-Banach theorem, there exists a linear continuous form L on $L^\infty(\mathbb{R})$ such that $L(u) = u(0)$ whenever $u \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. If we assume that for some $f \in L^1$, $L(u) = \int f u \, dx$ for all $u \in L^\infty$, then in particular, $\int f \varphi \, dx = 0$ for all continuous functions of compact support with $\varphi = 0$ near 0. This implies that $f = 0$ a.e., which is a contradiction.

Definition 1.2. The norm on B^* is given by

$$\|\xi\|_{B^*} = \sup_{0 \neq x \in B} \frac{|\langle x, \xi \rangle|}{\|x\|_B}.$$

Proposition 1.1. *For all $x \in B$,*

$$\|x\|_B = \sup_{0 \neq \xi \in B^*} \frac{|\langle x, \xi \rangle|}{\|\xi\|_{B^*}}.$$

Proof. We have $|\langle x, \xi \rangle| \leq \|x\|\|\xi\|$ by definition for all $\xi \in B^*$. So

$$\sup_{\xi \neq 0} \frac{|\langle x, \xi \rangle|}{\|\xi\|} \leq \|x\|.$$

On the other hand, let $W = \mathbb{C}x \subseteq B$, and let $\xi_0 : W \to \mathbb{C}$ be $\alpha x \mapsto \alpha\|x\|$. We have $|\xi_0(y)| = \|y\|$ for all $y \in W$, so by Hahn-Banach, ξ_0 extends to $\tilde{\xi} \in B^*$ such that $|\tilde{\xi}(y)| \leq \|y\|$ for all $y \in B$ and $\tilde{\xi}(x) = \|x\|$. So $\|\tilde{\xi}\| = 1$, which gives us

$$\|x\| = \frac{|\langle x, \tilde{\xi} \rangle|}{\|\tilde{\xi}\|} \leq \sup_{\xi \neq 0} \frac{|\langle x, \tilde{\xi} \rangle|}{\|\tilde{\xi}\|}.$$

□

Remark 1.2. This proposition implies that the natural map $\varphi : B \to B^{**}$ given by $x \mapsto (\xi \mapsto \langle x, \xi \rangle)$ is an isometry. The range is closed but may be strictly smaller than B^{**}.

1.2 Geometric version of the Hahn-Banach theorem

Definition 1.3. Let V be a normed vector space over $\mathbb{R}$. An **affine hyperplane** in V is a set of the form $H = f^{-1}(\alpha)$, where $\alpha \in \mathbb{R}$, f is linear, and $f \neq 0$.

Proposition 1.2. *The affine hyperplane $H = f^{-1}(\alpha)$ is closed if and only if f is continuous.*

Proof. It is clear that if f is continuous, then H is closed. Conversely, if H is closed, let $x_0 \in H^c$, which is open. We may assume that $f(x_0) < \alpha$. Let $r > 0$ be such that $B(x_0, r) = \{x \in V : \|x - x_0\| < r\} \cap H = \varnothing$.

We claim that $f(x) < \alpha$ for all $x \in B(x_0, r)$. If $f(x_1) > \alpha$ for some $x_1 \in B(x_0, r)$, then the line segment $\{tx_0 + (1-t)x_1 : 0 \leq t \leq 1\} \subseteq B(x_0, t)$, so $f(tx_0 + (1-t)x_1) \neq \alpha$ for all t. If $t = \frac{\alpha - f(x_0)}{f(x_1) - f(x_0)} \in (0, 1)$, we get a contradiction.

We get $f(x_0 + ry) < \alpha$ for all y with $\|y\| = 1$. So f is bounded, and hence f is continuous. □

Definition 1.4. Let V be a normed vector space over $\mathbb{R}$, and let $A, B \subseteq V$. We say that the affine hyperplane $H = f^{-1}(\alpha)$ **separates** A and B if we have $f(x) \leq \alpha$ for all $x \in A$ and $f(x) \geq \alpha$ for all $x \in B$.

Theorem 1.1 (geometric Hahn-Banach). *Let V be a normed vector space over $\mathbb{R}$, and let $A, B \subseteq V$ be convex, disjoint, and nonempty. Assume also that A is open. Then there exists a closed affine hyperplane separating A and B.*

This is sometimes called the "seperation theorem." We will prove this next time. Here is the idea of the proof. Given an open convex set $C \subseteq V$, define the gauge of C as $p(x) = \inf\{t > 0 : x/t \in C\}$.